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Photon tunneling and light transmission in transparent media

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Abstract : We show that the light transmission in transparent media can be described using photon tunneling phenomenon. Photon tunneling time for the index of refraction barrier can be investigated using Helmholtz wave equation. We report an analytical expression for the tunneling time of a photon from the index of refraction barrier as well as the transmission coefficient. It is predicted that the tunneling time depends on the refractive index, the thickness of barrier, and also on the incident wave frequency. We will exploit this idea to describe the light transmission in transparent media. A comparison with the experimental data and the theoretical result is also reported.

Keywords : Photon tunneling, transit time, light transmission in transparent media, index of refraction barrier.

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1. Introduction

In recent years, the tunneling time for a particle passing through a classically opaque potential barrier has been one of the most striking subjects. Most works in this field are based on the numerical and analytical analysis of the Schrödinger equation with some initial condition on the wavepackets [1–3], and have been confirmed experimentally [4–7]. The tunneling time of a particle can be obtained in different ways [3]. The analogy between quantum tunneling of a particle and an evanescent electromagnetic wave found in a low-dielectric constant region, separating two regions of high dielectric constant, can be considered and one can find a suitable approach for tunneling time of photon from this classically forbidden region. It has been shown that while the tunneling time of an electron is determined by solving the Schrödinger equation for potential barrier, the Helmholtz wave equation could be used for a photon tunneling. Therefore, the concept of potential energy in the particle tunneling changes to the index of refraction in the photon tunneling [8–10]. Therefore, we assume the medium to be lossless, non-dispersive and linear. In this work, we consider an analytical solution to the Helmholtz wave equation and derive an expression for the tunneling time. It has been shown that the tunneling time depends on the refractive index, the width of barrier, and also the frequency of incident light upon the barrier. Finally, we describe the light transmission in transparent media based on the tunneling idea. We consider the one dimensional crystal with periodic index of refraction profile. These types of structures, in general case, can be modelled and investigated using the standard approaches such as density matrix formalism. But in our case, because of our prediction about photon tunneling time from each layer, we can approximately accept the total transit time to be equal to the superposition of the individual tunneling times.

The organization of this paper is as follows. In Section 2, the basic idea for the tunneling time and the transmission coefficient through the index of refraction barrier will be discussed. Photonic treatment for describing

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of light transmission in transparent media presented in Section 3. Finally, the paper is ended with a conclusion, including the discussion about simulated results.

2. Photon tunneling principles

In quantum mechanics, by solving the Schrödinger equation, we can obtain the probability for finding an electron in a given position and time. As an example, we consider the potential barrier with electron energy lower than the magnitude of potential. When the width of these potential is smaller, we can obtain the small but nonzero value for electron probability in the other side. This phenomenon known as tunneling process has been discussed so far in the papers and text books for electron and particles. Now, we will use this idea and apply to photon case. The Maxwell equation in general and the Helmholtz equation in special case, can be equivalent to the Schrödinger equation in photon case. This idea is very important if we consider the photon and electron from the mathematical and the physical points of view. Thus, the electric field can give us the probability for finding photon in given position and time. The mediums in which the electromagnetic waves can not be propagated, are known as forbidden regions for photon or evanescent waves. In optical fiber communication, we use from this idea for restriction of light to core region in classical treatment. But, when this forbidden region have low width, the photon can be detected in the other side as an electron which is a typical problem in optical integrated circuits and optical linear and nonlinear couplers. So, for quantitative investigation of these phenomenon, we will present the analytical treatment for photon tunneling and probability of photon finding from the index of refraction barriers. For photon case, which can be described by wavepacket, has been considered for simplicity, as the plane wave with wavevector K and at the end, we consider the group velocity for calculation of tunneling time and any other necessary quantities. We study one-dimensional problem in which wavepacket propagates along one direction, say the z-axis. Let us consider a wavepacket moving along the z-axis incident on a region $[0,a]$ with a refractive index $n(z)$, as depicted in Figure 1. For simplicity, the barrier can be approximated as a square one, in which $n(z)$ is constant in regions I, II and III. For the sake of simplicity, we also assume that $n(z)$ is identical in regions I and III. We can take the component of the electric and magnetic fields as E and B respectively. As in general, we describe the propagation of light wave through the barrier by a scalar field Ψ, similar to the Schrödinger wave function in particle case. In order to obtain the tunneling time for the particle, one could solve the Schrödinger equation of motion. But here, using the physical similarities between electron and photon particles in one hand and the Schrödinger and Helmholtz equations on the other hand, we consider the Helmholtz equation [11–14]

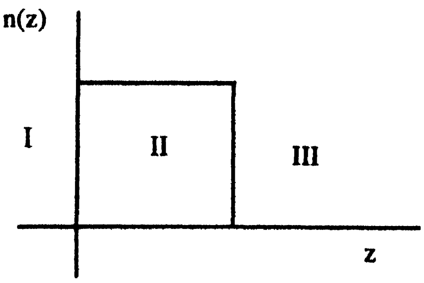

Figure 1. Index of refraction profile.

$$\nabla^2\Psi + K^2\Psi = 0 \cdot \tag{1}$$

Eq. (1) can be solved and using boundary conditions, all the constants can be found. The details of the calculation are given in Appendix A. Now, the transmission coefficient is given as

$$T = \frac{-4i\eta k_0}{(\eta - ik_0)^2}\,\frac{e^{-\eta a}}{1-e^{-2\eta a}r^2}\,, \tag{2}$$

where $k_0 = \dfrac{\omega}{c}n_0$ and $k = \dfrac{\omega}{c}n = i\eta$ are the wavevectors in regions I (or III) and II respectively and $r = \left(\dfrac{\eta + ik_0}{\eta - ik_0}\right.$

By substituting $-ik = \eta$ in eq. (2), the equation can be separated into real and imaginary parts and after finding the phase difference of transmission coefficient during the index of refraction barrier, we can obtain the tunneling time. There are many different ways to calculate the tunneling time [3,9,15–17]. Here, we consider the transit time τ for a wavepacket propagating through a given region measured at the interval between the arrival time of the signal envelope at the two ends of that region whose distance is a. In general, the wavepacket has a group velocity v_g, which means that $\tau = \dfrac{a}{v_g}$. Since $v_g = \dfrac{d\omega}{dk}$ (k is the wave vector and ω is the angular frequency), we can write [9]

$$\tau = \frac{d\phi}{d\omega}, \tag{3}$$

where $d\phi = adk$ is the phase difference of wavepacket in that region. Now, using eqs. (2) and (3), $k = \frac{\omega}{c} n$ and $r^2 = \left(\frac{n-n_0}{n+n_0}\right)^2$, we obtain the transit time as

$$\tau = \frac{na}{d\omega} \frac{(1-r^4)}{1+r^4-2r^2\cos 2ka} \tag{4}$$

where c is the speed of light in vacuum, n is the refractive index of region II, and a is the barrier width. Figure 2 shows the variation of tunneling time *versus* the barrier length and the incident wavelength.

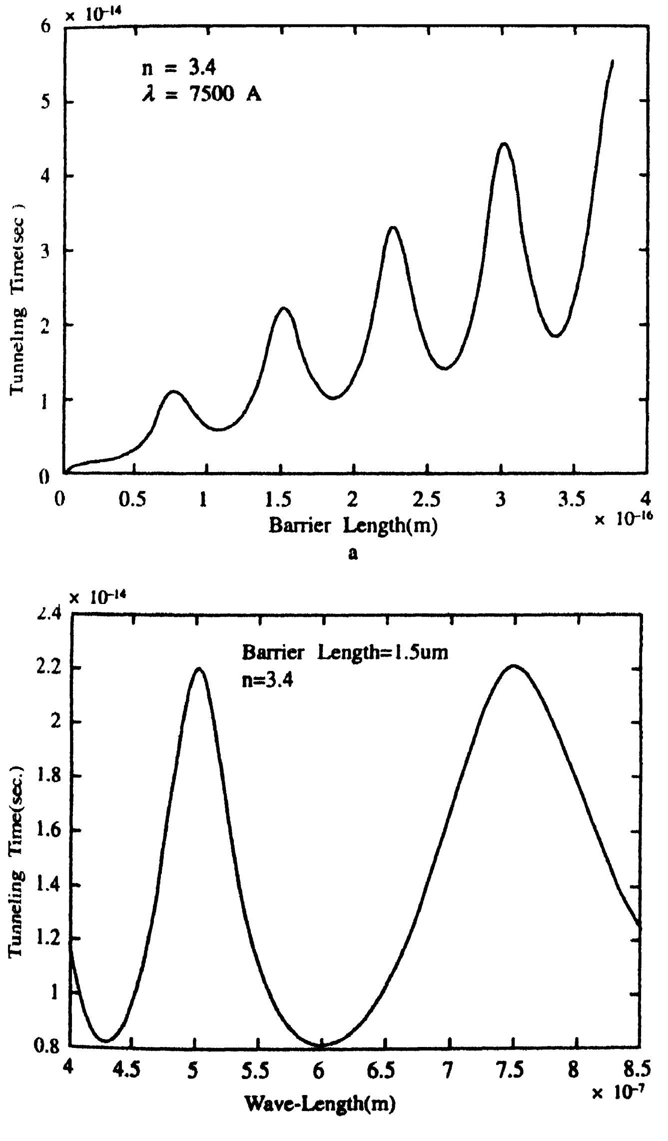

Figure 2. Tunneling time *versus*. barrier length and wave-length in macroscopic scale (a) tunneling time–barrier length, (b) tunneling time–wave-length.

3. Photonic treatment

In this section, we will present the tunneling based physical description for the light transmission in transparent media. For simplicity, we restrict ourself to the light propagation in one dimensional case. In this case, the index of refraction is a periodic function. So, it is necessary to exactly determine the index of refraction barrier pattern for this case which have atoms and free space between the atoms. For free space, the index of refraction is unity and for atoms we calculate the index of refraction quantum mechanically. For this purpose, one should calculate the dielectric constant or optical susceptibility. Here, we calculate the optical susceptibility by utilizing the density matrix formulation [14]. Let us first confine ourselves to the case of an isotropic material. As a consequence of symmetry considerations, polarization vector P is parallel to the electric field E in such medium. For the case of an external incident monochromatic plane waves of frequency ω, the steady state response of the medium is also monochromatic with the same frequency ω, so that we may write

$$E(r,t) = E(r)e^{-i\omega t} \tag{5}$$

$$P(r,t) = P(r)e^{-i\omega t} \tag{6}$$

Therefore we can express the linear susceptibility as a scalar quantity $\chi(\omega)$ defined by $P(\omega) = \chi(\omega)E(\omega)$.

The density matrix formulation gives the susceptibility $\chi(\omega)$ as follows [14,15]

$$\chi(\omega) = \sum_n \frac{N|\mu_{nm}|^2}{3\hbar}\left[\frac{1}{(\omega_{nm}-\omega)-i\gamma_{nm}} + \frac{1}{(\omega_{nm}+\omega)+i\gamma_{nm}}\right] \tag{7}$$

where ω_{nm} and μ_{nm} are the resonant frequency and the dipole matrix element vector for the atomic transition respectively, and N is the atomic number density in the medium. The damping rates γ_{nm} are given by

$$= \frac{1}{2}(\Gamma_n + \Gamma_m) + \gamma_{nm}^{col.}, \tag{8}$$

where Γ_n and Γ_m denote the total rate of population out of levels n and m respectively. The quantity γ_{nm} is the dipole-dephasing rate due to processes like elastic collision. If we use the rotating wave approximation and ignore the effects of dephasing rate, the eq. (7) can be written [15] as

$$\chi(\omega)=\frac{N|\mu_{nm}|^2}{3\hbar(\omega_{nm}-\omega)} \qquad (9)$$

The refractive index $n(\omega)$ of the medium is related to the linear dielectric constant $\varepsilon(\omega)$ and linear susceptibility $\chi(\omega)$ through

V(z)

0 a b z

Figure 3. A periodic index of refraction in one dimensional crystal.

$$n(\omega)=\sqrt{\varepsilon(\omega)}=\sqrt{1+4\pi\chi(\omega)}=\left(1+\frac{4\pi|\mu_n}{(\omega_{nm}-\omega)}\right)^{1/2}. \qquad (10)$$

We have ignored the local field effects on atomic polarization. By using eq. (10), we try to obtain the refractive index for single atom in one-dimensional crystal. Now, in this case, we have the periodic index of refraction as shown in Figure 3, where a and b are the atomic diameter and free space between two atoms, respectively. In order to obtain the total transit time, we must calculate it for this periodic structure. Using the basic methods in layered media such as density matrix formalism, we can obtain the transmission coefficient for this periodic structure. But, because of our prediction about tunneling time from one barrier which is approximately about 10^{-18} sec, we can accept the tunneling to be a sequential process, and the total transit time can be obtained by summation of the individual tunneling times. For numerical calculation, we consider the total transit time in diamond crystal such as Si and Ge in one dimensional approximation. The considered material plays important role in integrated electronics and have applications in optical tunneling devices. The numerical results show that for one centimeter length, the total transit time alter in an interval between 10^{-11}s and 10^{-10}s. These values correspond to maximum and minimum numbers of atoms at any crystal directions ((100), (111)). These values have a good agreement with classical prediction [16,17]. Also, in the atomic scale, eq. (4) shows that the tunneling time is of the order of $\tau \approx 10^{-19}$–10^{-18} sec which have good compatibility with [18]. Table 1 shows compatibility between our theoretical result given in eq. (4) with the reported experimental results [19,20]. According to the reported result [19] with two fused silica prisms with n_1 = 1.403 and air gap n_2 = 1, and Gaussian laser beam of wavelength 3.39 μm with an incident angle about 45.5 degree, the measured tunneling time is 40 fsec. Our theoretical result predicts this to be 36.8 fsec. For another confirmation, let us consider the case with two paraffin prisms n_1 = 1.49 and and air gap n_2 = 1 with incident angle about 60 degrees. The measured value is about 87 psec [19], while the predicted result is about 87.2 psec. Thus it is also evident that the theory matches quite well with the experiments. Finally, we consider the recent experiment [19] with two perspex prisms n_1 = 1.605 separated by an air gap n_2 = 1. For an incident angle of 45 degree, the reported result is 117 psec and our predicted value is 81 psec. Other reported result and our theoretical prediction are given in Table 1. In this table, there is a deviation between experiment and theoretical result in number 3. Probably, this is due to the fact that the condition of opaque barrier is not completely fulfilled

Table 1. Comparison of experimental and theoretical result.

Theory	Experiment
36.8 fsec	40 fsec [18]
87.2 psec	87 ± 7 psec [18]
81 psec	117 ± 10 psec [18]
128 psec	130 psec [19]
2.81 fsec	2.71 fsec [19]

4. Conclusion

The general behaviour of the tunneling time through a barrier is shown in Figure 2. From Figure 2, we see that with the increas of (a) barrier length and (b) wavelength, the tunneling time have oscillations. In this paper, we obtained the explicit formula for the tunneling time of the refractive index barrier. Eq.(4) shows that the tunneling time τ explicitly depends on the refractive index, the barrier width and the incident wave frequency. In the case of $r \ll 1$, tunneling time approaches to na/c. This means that by increasing barrier width, the tunneling time is increased. In other words, τ and a are linearly dependent and results are shown in Figure 4. Also the tunneling time *versus* the refractive index of the medium is displayed in Figure 5. In Figure 6 variations of the

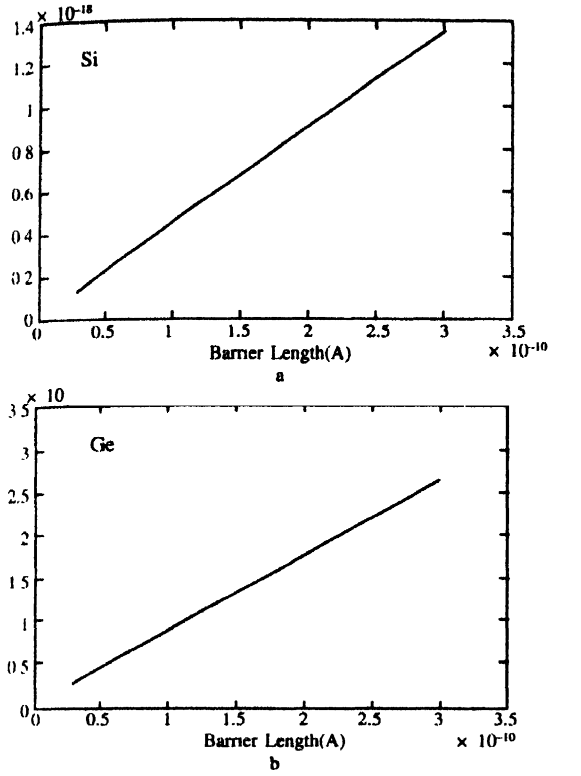

Figure 4. Tunneling time *versus* barrier length : (a) Si (wave-length = 7500A), (b) Ge (wave-length = 4500A).

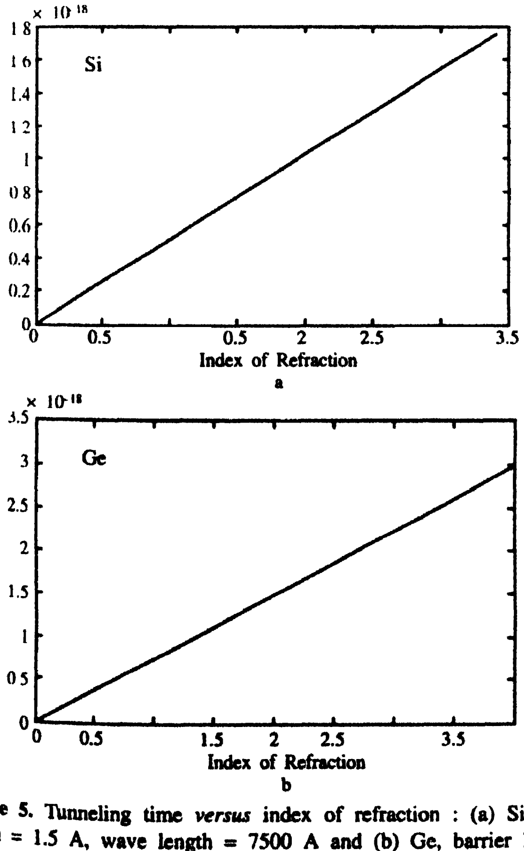

Figure 5. Tunneling time *versus* index of refraction : (a) Si, barrier length = 1.5 A, wave length = 7500 A and (b) Ge, barrier length = 1.75 A, wave length = 4500 A.

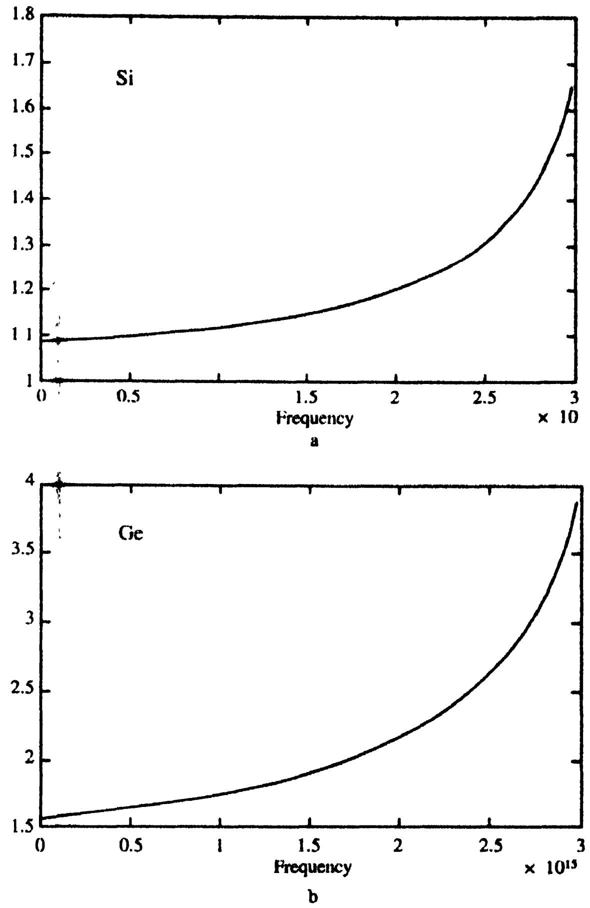

Figure 6. Index of refraction *versus* frequency. (a) Si, (b) Ge.

refractive index due to incident frequency for Si and Ge are shown. Numerical results of our model and classical limit were presented, and as it can be seen, have a good compatibility. Table 1 show the compatibility between experimental data and our theoretical results.

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Appendix A

Here, we present a simple proof for the eq. (2). By matching boundary conditions in $z = 0$ and $z = a$ for the following wave-functions of eq. (1), we have

$$\Psi_I = e^{ik_0 z} + \mathrm{R}e^{-ik_0 z}$$
$$\Psi_{II} = Ae^{-\eta z} + Be^{\eta z}$$
$$\Psi_{III} = Te^{ik_0(z-a)}$$

$$1 + R = A + B \tag{A-1}$$

$$ik_0 - ik_0 R = -A\eta + B\eta \tag{A-2}$$

$$Ae^{-\eta a} + Be^{\eta a} = T \tag{A-3}$$

$$-A\eta e^{-\eta a} + B\eta e^{\eta a} = ik_0 T \tag{A-4}$$

With some mathematical manipulation we can obtain the following relation for A and B

$$A = \frac{T(\eta + ik_0)e^{\eta a} - 2ik_0 e^{2\eta a}}{(\eta + ik_0) + (\eta - ik_0)e^{2\eta a}}, \tag{A-5}$$

$$B = \frac{2ik_0 + (\eta - ik_0)e^{\eta a}.T}{(\eta + ik_0) + (\eta - ik_0)e^{2\eta a}}. \tag{A-6}$$

Now, by substituting the eqs. (A-5,6) into eq. (A-4), we obtain the exact form for T

$$T = \frac{\frac{-4ik_0\eta}{(\eta - ik_0)^2} e^{-\eta a}}{1 - \left(\frac{\eta + ik_0}{\eta - ik_0}\right)^2 e^{-2\eta a}} \tag{A-7}$$

If we define $r = \frac{\eta + ik_0}{\eta - ik_0}$, one can obtain the following relation

$$1 - r^2 = -\frac{4ik_0\eta}{(\eta - ik_0)^2}.$$

So, the final form for transmission coefficient (eq. (2)) can be given as

$$T = \frac{(1 - r^2)e^{-\eta a}}{1 - r^2 e^{-2\eta a}} \tag{A-8}$$

Also, by defining $k = i\eta$ and by inserting this definition into eq. (A-8), we obtain the following form for T as

$$T = \frac{1 - r^2}{(1 - r^2\cos 2ka)^2 + (r^2 \sin 2ka)^2}$$
$$[\cos ka(1 - r^2\cos 2ka) - r^2 \sin ka \sin 2ka$$
$$+ i(\sin ka(1 - r^2 \cos 2ka) + r^2 \sin 2ka \cos ka)] \tag{A-9}$$

Thus, the phase of transmission coefficient is

$$\phi = \tan^{-1}\frac{r^2\cos ka \sin 2ka + \sin ka(1 - r^2\cos 2ka)}{\cos ka(1 - r^2\cos 2ka) - r^2 \sin ka \sin 2ka} \tag{A-10}$$

Using the basic definition presented in eq. (3), we obtain the final relation for tunneling time as

$$\tau = \frac{\frac{\eta a}{}(1 - r^4)}{1 + r^2 - 2r^2\cos 2ka} \tag{A-11}$$

which is in complete agreement with the relation obtained in eq. (4).